Section de mathématiques

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Functional analysis 2
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Sheet 14

Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 14.1 (Quadratic perturbations of convex functionals)
Let $E\colon H \to \mathbb{R}_+ \cup \{\infty\}$ be a convex functional on a real Hilbert space H. Given any $x \in H$ and any $\tau > 0$, we define the convex functional $F_{x,\tau}\colon H \to \mathbb{R}_+ \cup \{\infty\}$ by

$$F_{x,\tau}(y) := E(y) + \frac{\|y-x\|^2}{2\tau}.$$

Show $\mathscr{D}(F_{x,\tau}) = \mathscr{D}(E)$ and for every $y \in H$ that

$$\partial^- F_{x,\tau}(y) = \partial^- E(y) + \frac{y-x}{\tau}.$$

where we use the convention that translates of the empty set are empty.

Solution 14.1 : It is clear that E is real-valued if and only if $F_{x,\tau}$ is, whece $\mathscr{D}(F_{x,\tau}) = \mathscr{D}(E)$. Now we prove the claimed inequality between subdifferentials. Let $y \in \mathscr{D}(E)$. Since the operator $\|\cdot - x\|^2/2\tau$ is differentiable at y with differential $\langle y - x, \cdot\rangle/\tau$, we obtain

$$\partial^- \frac{\|\cdot - x\|^2}{2\tau}(y) = \Big\{\frac{y-x}{\tau}\Big\}.$$

By the definition of subdifferential, we directly get

$$\partial^- E(y) + \frac{y-x}{\tau} = \partial^- E(y) + \partial^- \frac{\|\cdot - x\|^2}{2\tau}(y) \subset \partial^- F_{x,\tau}(y).$$

Conversely, let $y^* \in \partial^- F_{x,\tau}(y)$. Then for every $z \in H$ and every $\varepsilon \in (0,1)$,

$$E(y) + \frac{\|y-x\|^2}{2\tau} - \langle y^*, \varepsilon\, z\rangle \le E(y - \varepsilon z) + \frac{\|y - x - \varepsilon\, z\|^2}{2\tau}.$$

There is nothing to show if $E(y - \varepsilon\, z) = \infty$ for every $\varepsilon \in (0,1)$. If there exists $\varepsilon' \in (0,1)$ such that $E(y - \varepsilon'\, z) < \infty$, then $E(y - \varepsilon\, z) < \infty$ for every $\varepsilon \in (0,1)$ by convexity of E. For such ε, the above inequality rewrites as

$$\frac{E(y) - E(y - \varepsilon\, z)}{\varepsilon} \le \langle y^*, z\rangle + \frac{\|y - x - \varepsilon\, z\|^2 - \|y-x\|^2}{2\tau\,\varepsilon}.$$

Using convexity of E in the left-hand side, we get the desired inequality

$$\begin{aligned} E(y) - E(y - z) &\le \liminf_{\varepsilon \to 0+} \frac{E(y) - E(y - \varepsilon\, z)}{\varepsilon} \\ &\le \langle y^*, z\rangle + \liminf_{\varepsilon \to 0+} \frac{\|y - x - \varepsilon\, z\|^2 - \|y-x\|^2}{2\tau\,\varepsilon} \\ &= \Big\langle y^* - \frac{y-x}{\tau}, z\Big\rangle. \end{aligned}$$

Ex 14.2 (Uniqueness and fundamental properties of gradient flow trajectories*)
Let $x\colon \mathbb{R}_+ \to H$ be a gradient flow trajectory of a functional E as specified in the lecture notes. Show the following properties.

a) Given any other gradient flow trajectory $y\colon \mathbb{R}_+ \to H$, every $t \in \mathbb{R}_+$ satisfies

$$\big\|x(t) - y(t)\big\| \leq \big\|x(0) - y(0)\big\|.$$

In particular, gradient flow trajectories with fixed initial points are unique.
Hint: Differentiate the assignment $t \mapsto \|x(t) - y(t)\|^2/2$.

b) The assignment $t \mapsto E(x(t))$ is nondecreasing on $\mathbb{R}_+$ and locally Lipschitz continuous on $(0, \infty)$.
Hint: First prove local Lipschitz continuity of x on $(0, \infty)$. To this aim, it suffices to prove $\|\dot{x}\|$ is bounded on each interval of the form $[\varepsilon, \infty)$, where $\varepsilon > 0$. Deduce local Lipschitz continuity of $E \circ x$ using the defining properties of a gradient flow trajectory.

c) For every $z \in H$ and every $t > 0$,

$$E(x_t) \leq E(z) + \frac{\big\|x(0) - z\big\|^2}{2t}.$$

Hint: Use a) and b).

Ex 14.3 (Slope and Laplacian)
Let $E\colon H \to \mathbb{R}_+ \cup \{\infty\}$ be a convex and lower semicontinuous functional with nonempty domain. We define its slope $|\partial^- E|\colon H \to \mathbb{R}_+ \cup \{\infty\}$ through

$$|\partial^- E|(x) := \begin{cases} \sup\limits_{y \in H \setminus \{x\}} \dfrac{(E(y) - E(x))^-}{\|x - y\|} & \text{if } x \in \mathscr{D}(E), \\ \infty & \text{otherwise.} \end{cases}$$

a) Show for every $x \in \mathscr{D}(\partial^- E)$ and every $x^* \in \partial^- E(x)$,

$$|\partial^- E|(x) \leq \|x^*\|.$$

b) Deduce that for every $x \in \mathscr{D}(\partial^- E)$,

$$|\partial^- E|(x) = \min_{x^* \in \partial^- E(x)} \|x^*\|.$$

Hint: Given any $\tau > 0$, consider the unique minimizer $x_\tau \in H$ of the functional $F_{x,\tau}$ from above. You can use without proof that $\|x - x_\tau\|/\tau \leq |\partial^- E|(x)$.

c) Let $x\colon \mathbb{R}_+ \to H$ be a gradient flow trajectory of E. Deduce from b) that for *every* $t > 0$, the right derivative

$$x'^+(t) := \lim_{h \to 0+} \frac{x(t+h) - x(t)}{h}$$

exists in H. Moreover, it is equal to minus the unique element of minimal norm in $\partial^- E(x_t)$. The same holds at zero if $x(0) \in \mathscr{D}(\partial^- E)$.
Hint: First clarify why it suffices to prove the claim for $t = 0$ assuming $|\partial^- E|(x(0)) < \infty$. Then show for every $h > 0$ that $\|x(h) - x(0)\|/h \leq |\partial^- E|(x(0))$.

Solution 14.3 : (a) For every $y \in H$, the inclusion $x^* \in \partial^- E(x)$ entails

$$E(x) + \langle x^*, y - x\rangle \leq E(y).$$

Rearranging and applying the Cauchy–Schwarz inquality, we get

$$E(x) - E(y) \le \|x^*\|\,\|x-y\|$$

and consequently

$$(E(x) - E(y))^+ \le \|x^*\|\,\|x-y\|.$$

This gives the desired inequality by definition of $|\partial^- E|(x)$.
(b) By (a), we have the inequality "$\le$". To prove the reverse inequality, we use the hint $|\partial^- E|(x) \ge \|x - x_\tau\|/\tau$. By the first clause, $|\partial^- E|(x)$ is finite. Therefore, by the Banach–Alaoglu–Bourbaki theorem, there exist a sequence $(\tau_n)_{n\in\mathbb{N}}$ decreasing to zero and $v \in H$ such that $(x - x_{\tau_n})/\tau_n \rightharpoonup v$ weakly in H as $n \to \infty$. Since $(x - x_{\tau_n})/\tau_n \in \partial^- E(x_{\tau_n})$ by Proposition 5.9 from the lecture notes, we deduce from the strong-weak closure of Proposition 5.4 ibid. that $v \in \partial^- E(x)$. Since

$$\|v\| \le \liminf_{n\to\infty} \frac{\|x - x_{\tau_n}\|}{\tau_n} \le |\partial^- E|(x), \tag{1}$$

the desired reverse inequality is established.
(c) It suffices to restrict ourselves to $t = 0$ since for every $t \in \mathbb{R}_+$, the curve $y\colon \mathbb{R}_+ \to H$ given by $y(s) := x(t+s)$ is the unique gradient flow trajectory starting at $x(0)$. Moreover, if $|\partial^- E|(x(0)) = \infty$ then $\partial^- E(x(0))$ is necessarily empty by (b), which cannot happen in the interior of a gradient flow trajectory by definition (or at the initial point under the additional assumption $\partial^- E(x(0)) \neq \emptyset$). The other inequality from the hint from (c) is a simple consequence of the hint from (b) and the fact that the minimizing movement scheme converges uniformly to the gradient flow trajectory x, as shown in the lecture. As in (b), this implies the existence of a sequence $(h_n)_{n\in\mathbb{N}}$ decreasing to zero and $v \in H$ such that $(x_{h_n} - x)/h \rightharpoonup v$. As for (1), we deduce $\|v\| \le |\partial^- E|(x(0))$. The claim follows from (b) if we prove $v \in \partial^- E(x(0))$ since there exists precisely one element in $\partial^- E(x(0))$ with minimal norm by Remark 5.10. For $y \in H$, continuity of the scalar product with respect to the limit process in the definition of Bochner integrals and the fundamental theorem of calculus yield

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{h_n}\int_0^{h_n} \big\langle x'(t), x(t) - y\big\rangle \,\mathrm{d}t &= \lim_{n\to\infty} \Big[\Big\langle \frac{1}{h_n}\int_0^{h_n} x'(t)\,\mathrm{d}t, x(0) - y\Big\rangle \\ &\qquad + \frac{1}{h_n}\int_0^{h_n} \big\langle x'(t), x(t) - x(0)\big\rangle \,\mathrm{d}t\Big] \\ &= \langle v, x(0) - y\rangle.\end{aligned}$$

Combined with lower semicontinuity of E, and the inclusion $-x'(t) \in \partial^- E(x(t))$ for every $t > 0$, we deduce

$$E(x(0)) + \langle v, x(0) - y\rangle \le \lim_{n\to\infty} \frac{1}{h_n}\int_0^{h_n} \big[E(x(t)) + \big\langle x'(t), x(t) - y\big\rangle \,\mathrm{d}t\big] \le E(y).$$

This shows $v \in \partial^- E(x(0))$, as desired.

Ex 14.4 (Extension to the closure of the domain)
The goal of this exercise is to prove that gradient flow trajectories can be started from any point in the closure of the domain (and not only the domain, as already covered in the lecture). To this aim, since $o \in \overline{\mathscr{D}(E)}$, fix a sequence $(o_n)_{n\in\mathbb{N}}$ in $\mathscr{D}(E)$ converging to o. We aim to show the corresponding gradient flow trajectories starting at the o_n's converge to a gradient flow trajectory starting at o.

a) Let $x^n\colon \mathbb{R}_+ \to H$ denote the gradient flow trajectory starting at o_n, where $n \in \mathbb{N}$. Show $(x^n)_{n\in\mathbb{N}}$ converges uniformly to a continuous curve $x\colon \mathbb{R}_+ \to H$.
Hint: Use Exercise 14.2.

b) Given any $t_0 > 0$, show the existence of a constant $C(t_0) > 0$ such that for every $n \in \mathbb{N}$,

$$\frac{1}{2}\int_{t_0}^{\infty} \big\|\dot{x}^n(t)\big\|^2 \,\mathrm{d}t \leq C(t_0)$$

Hint: Use Exercise 14.2 to find a uniform upper bound on $E(x^n_{t_0})$. Then apply the energy estimate from the proof of Theorem 5.11 from the lecture notes.

c) Show the curve x from a) is a gradient flow trajectory starting at o.

Solution 14.4 : (a) From the contraction property in Exercise 14.2, for every $n, m \in \mathbb{N}$ we get

$$\sup_{t\in\mathbb{R}_+} \big\|x^n(t) - x^m(t)\big\| \leq \|o_n - o_m\|.$$

Since $(o_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H, this estimate shows $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space of H-valued continuous maps defined on $\mathbb{R}_+$ with respect to the supremum norm. This space is complete. Therefore, $(x^n)_{n\in\mathbb{N}}$ converges uniformly to a continuous limit $x\colon \mathbb{R}_+ \to H$.
(b) By the proof of Theorem 5.11 for the construction of x^n for every $n \in \mathbb{N}$, given any $t_0 > 0$ we infer the inequality

$$\frac{1}{2}\int_{t_0}^{\infty} \big\|\dot{x}^n(t)\big\|^2 \,\mathrm{d}t \leq E(x^n(t_0)).$$

Thus, it suffices to find an upper bound on the right-hand side which is uniform in $n \in \mathbb{N}$. To this aim, we observe for every $t > 0$ and every $y \in H$,

$$\begin{aligned}
t\,E(x^n(t)) &\leq \int_0^t E(x^n(s))\,\mathrm{d}s \\
&\leq t\,E(y) - \int_0^t \big\langle \dot{x}^n(s), x^n(s) - y\big\rangle \,\mathrm{d}s \\
&= t\,E(y) - \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\frac{\big\|x^n(s) - y\big\|^2}{2}\,\mathrm{d}s \\
&= t\,E(y) - \frac{\big\|x^n(t) - y\big\|^2}{2} + \frac{\big\|o_n - y\big\|^2}{2} \\
&\leq t\,E(y) + \frac{\big\|o_n - y\big\|^2}{2}.
\end{aligned}$$

Setting $t = t_0$ and choosing $y \in \mathscr{D}(E)$, this gives the desired bound on $\sup_{n\in\mathbb{N}} E(x^n(t_0))$.
(c) The uniform bound provided by (b) implies x^n is 2-absolutely continuous on $[t_0, \infty)$ for every $n \in \mathbb{N}$ with uniformly bounded energy. Arguing as in the proof of Theorem 5.11, we infer $\dot{x}^n \rightharpoonup \dot{x}$ weakly in $L^2([t_0,\infty); H)$. To show x is a gradient flow trajectory, it then suffices to pass to the limit as $n \to \infty$ in the following inequality for every $y \in H$ and every $s, t \in [t_0, \infty)$ with $s < t$, which holds for every $n \in \mathbb{N}$ by the proof of Theorem 5.11 :

$$\int_s^t E(x^n(r)) + \big\langle \dot{x}^n(r), x^n(r) - y\big\rangle \,\mathrm{d}r \leq (t-s)\,E(y).$$